10.14 LAPLACE TRANSFORM OF ERROR FUNCTION

Example 29. *Find* $L\{erf\sqrt{t}\}$ *and hence prove that*

$$L\{t.erf\,2\sqrt{t}\} = \frac{3s+8}{s^2(s+4)^{3/2}}$$

(*U.P. II Semester, Summer 2001*)

Solution. We know that $erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\dots\right)dx = \frac{2}{\sqrt{\pi}}\left[x-\frac{x^3}{3}+\frac{x^5}{10}-\frac{x^7}{42}\dots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}}\left[\sqrt{t}-\frac{t^{3/2}}{3}+\frac{t^{5/2}}{10}-\frac{t^{7/2}}{42}+\dots\right]$$

$$L\{erf\sqrt{t}\} = \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}}-\frac{\Gamma\left(\frac{5}{2}\right)}{3s^{5/2}}+\frac{\Gamma\left(\frac{7}{2}\right)}{10s^{7/2}}-\frac{\Gamma\left(\frac{9}{2}\right)}{42s^{9/2}}+\dots\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}}-\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{3s^{5/2}}+\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{10s^{7/2}}-\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{42s^{9/2}}+\dots\right] \quad \left[\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$$

$$= \frac{1}{s^{3/2}}-\frac{1}{2}\frac{1}{s^{5/2}}+\frac{1.3}{2.4}\frac{1}{s^{7/2}}-\frac{1}{2}.\frac{3}{4}.\frac{5}{6}\frac{1}{s^{9/2}}+\dots$$

$$= \frac{1}{s^{3/2}}\left[1-\frac{1}{2}.\frac{1}{s}+\frac{1}{2}.\frac{3}{4}.\frac{1}{s^2}-\frac{1}{2}.\frac{3}{4}.\frac{5}{6}\frac{1}{s^3}+\dots\right]$$

$$= \frac{1}{s^{3/2}}\left[1-\frac{1}{2}.\frac{1}{s}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\frac{1}{s^2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\frac{1}{s^3}+\dots\right]$$

$$= \frac{1}{s^{3/2}}\left[1+\frac{1}{s}\right]^{-\frac{1}{2}} = \frac{1}{s^{3/2}}\left[\frac{s}{s+1}\right]^{\frac{1}{2}} = \frac{1}{s\sqrt{(s+1)}}$$ **Ans.**

Now, $L\{erf(2\sqrt{t})\} = L\{erf\sqrt{4t}\} = \frac{1}{4}\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} = \frac{2}{s\sqrt{s+4}}$

$$L\{t.erf(2\sqrt{t})\} = -\frac{d}{ds}\frac{2}{\sqrt{s^3+4s^2}} = -2\left(-\frac{1}{2}\right)\left[s^3+4s^2\right]^{-\frac{3}{2}}\left(3s^2+8s\right)$$

$$= \frac{3s^2+8s}{\left(s^3+4s^2\right)^{3/2}} = \frac{s(3s+8)}{s^3(s+4)^{3/2}} = \frac{3s+8}{s^2(s+4)^{3/2}}$$ **Proved.**

10.15 COMPLEMENTARY ERROR FUNCTION

This function is defined by

$$erf_c\left(\sqrt{t}\right) = 1-erf\left(\sqrt{t}\right)$$

$$= 1-\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$$

Now, $L\{erf_c(\sqrt{t})\} = L\left\{1-\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx\right\}$

$$= L(1)-\frac{2}{\sqrt{\pi}}L\left\{\int_0^{\sqrt{t}} e^{-x^2}\,dx\right\} = \frac{1}{s}-\frac{1}{s\sqrt{s+1}}$$

$$= \frac{\sqrt{(s+1)}-1}{s\sqrt{(s+1)}} = \frac{\left\{\sqrt{(s+1)}-1\right\}\left\{\sqrt{(s+1)}+1\right\}}{s\sqrt{(1+s)}\left\{\sqrt{(s+1)}+1\right\}}$$

$$= \frac{s+1-1}{s\sqrt{s+1}\,(\sqrt{s+1}+1)}$$

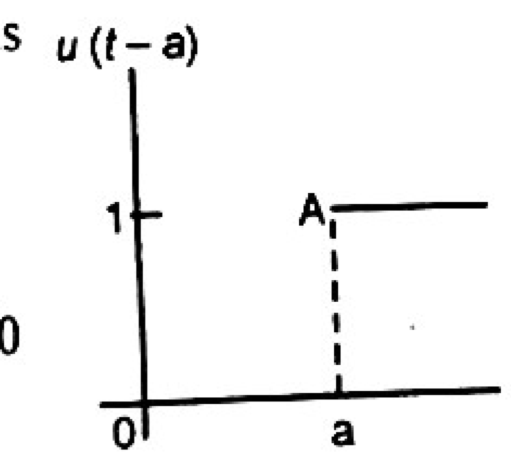
$$= \frac{1}{\sqrt{(s+1)}\left\{\sqrt{(s+1)}+1\right\}}$$

$$\therefore L\left[erf_c\,(\sqrt{t})\right] = \frac{1}{\sqrt{s+1}\left\{\sqrt{s+1}+1\right\}}$$ **Ans.**

10.16 UNIT STEP FUNCTION

With the help of unit step functions, we can find the inverse transform of functions, which cannot be determined with previous methods.

The unit step function $u(t-a)$ is defined as follows:



$$u(t-a) = \begin{cases} 0 \text{ when } t < a \\ 1 \text{ when } t \geq a \end{cases}$$ where $a \geq 0$

10.17 LAPLACE TRANSFORM OF UNIT FUNCTION

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

Proof. $L[u(t-a)] = \int_0^\infty e^{-st}u(t-a)\,dt$

$$= \int_0^a e^{-st}.0\,dt + \int_a^\infty e^{-st}.1\,dt = 0+\left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$L[u(t-a)] = \frac{e^{-as}}{s}$$ **Proved.**

Example 30. *Express the following function in terms of unit step functions and find its Laplace transform:*

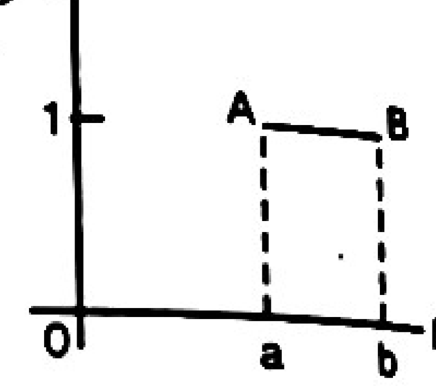
$$f(t)=\begin{cases}8, & t<2\\ 6, & t\ge 2\end{cases}$$

Solution.
$$f(t)=\begin{cases}8+0, & t<2\\ 8-2 & t\ge 2.\end{cases}$$
$$=8+\begin{cases}0, & t<2\\ -2, & t\ge 2\end{cases}$$
$$=8+(-2)\begin{cases}0, & t<2\\ 1, & t\ge 2\end{cases}$$
$$=8-2u(t-2)$$

$$L\{f(t)\}=8L(1)-2\,L\,u(t-2)=\frac{8}{s}-2\frac{e^{-2s}}{s}$$ **Ans.**

Example 31. *Draw the graph of $u(t-a)-u(t-b)$.*

Solution. As in Art 10.15 the graph of $u(t-a)$ is a straight line parallel to t-axis from A to ∞.



Similarly, the graph of $u(t-b)$ is a straight line parallel to t-axis from B to ∞.

Hence, the graph of $u(t-a)-u(t-b)$ is AB. **Ans.**

10.18 SECOND SHIFTING THEOREM

If $L[f(t)]=F(s)$, then $\boxed{L[f(t-a).u(t-a)]=e^{-as}F(s)}$

Proof.
$$L[f(t-a).u(t-a)]=\int_0^\infty e^{-st}[f(t-a).u(t-a)]dt$$
$$=\int_0^a e^{-st}f(t-a).0\,dt+\int_a^\infty e^{-st}f(t-a)(1)\,dt$$
$$=\int_a^\infty e^{-st}f(t-a)\,dt$$
$$=\int_0^\infty e^{-s(u+a)}f(u)\,du,\quad \text{where } u=t-a$$
$$=e^{-sa}\int_0^\infty e^{-su}.f(u)\,du=e^{-sa}F(s)$$ **Proved.**

Example 32. *Express the following function in terms of unit step function and find its Laplace transform:*

$$f(t)=\begin{cases}E, & a<t<b\\ 0, & t\ge b\end{cases}$$

Solution.
$$f(t)=E\begin{cases}1, & a<t<b\\ 0, & t\ge b\end{cases}$$
$$[L[f(t-a).u(t-a)]=e^{-as}F(s)]$$
$$L\{f(t)\}=E\left[\frac{e^{-as}}{s}-\frac{e^{-bs}}{s}\right]$$ **Ans.**

Example 33. *Express the following function in terms of unit step function:*

$$f(t)=\begin{cases}t-1, & 1<t<2\\ 3-t, & 2<t<3\end{cases}$$

and find its Laplace transform. (U.P.; II Semester, 2009)

Solution.
$$f(t)=\begin{cases}t-1, & 1<t<2\\ 3-t, & 2<t<3\end{cases}$$
$$=(t-1)[u(t-1)-u(t-2)]+(3-t)[u(t-2)-u(t-3)]$$
$$=(t-1)u(t-1)-(t-1)u(t-2)+(3-t)u(t-2)+(t-3)u(t-3)$$
$$=(t-1)u(t-1)-2(t-2)u(t-2)+(t-3)u(t-3)$$
$$=e^{-s}L(t)-2e^{-2s}L(t)-e^{-3s}L(t)$$
$$[L[f(t-a).u(t-a)]=e^{-as}F(s)]$$

$$L[f(t)]=\frac{e^{-s}}{s^2}-2\frac{e^{-2s}}{s^2}+\frac{e^{-3s}}{s^2}$$ **Ans.**

Example 34. *Find $L\{F(t)\}$ if*

$$F(t)=\begin{cases}\sin\left(t-\frac{\pi}{3}\right), & t>\frac{\pi}{3}\\ 0, & t<\frac{\pi}{3}\end{cases}$$

Solution.
$$L\{F(t)\}=e^{-s\frac{\pi}{3}}L(\sin t)\qquad \left[\because a=\frac{\pi}{3}\right]$$
$$=e^{-s\frac{\pi}{3}}.\frac{1}{s^2+1}$$ (Using second shifting property) **Ans.**

10.19 THEOREM. $L[f(t)u(t-a)]=e^{-as}L[f(t+a)]$

Proof.
$$L[f(t).u(t-a)]=\int_0^\infty e^{-st}[f(t).u(t-a)]dt$$
$$=\int_0^a e^{-st}[f(t).u(t-a)]dt+\int_a^\infty e^{-st}.[f(t).u(t-a)]dt$$
$$=0+\int_a^\infty e^{-st}.f(t)(1)\,dt$$
$$=\int_a^\infty e^{-s(y+a)}.f(y+a)\,dy$$
$$=e^{-as}\int_a^\infty e^{-sy}.f(y+a)\,dy\qquad (t-a=y)$$
$$=e^{-as}\int_a^\infty e^{-st}f(t+a)\,dt=e^{-as}L[f(t+a)]$$ **Proved.**

Example 35. *Find the Laplace transform of $t^2\,u(t-3)$.*

Solution.
$$t^2.u(t-3)=\left[(t-3)^2+6(t-3)+9\right]u(t-3)$$
$$=(t-3)^2.u(t-3)+6(t-3).u(t-3)+9u(t-3)$$

$$L[t^2.u(t-3)] = L[(t-3)^2.u(t-3)] + 6L[(t-3).u(t-3)] + 9L[u(t-3)]$$
$$= e^{-3s}\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right]$$

Aliter. $L[t^2u(t-3)] = e^{-3s}L(t+3)^2 = e^{-3s}L\left[t^2+6t+9\right] = e^{-3s}\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right]$ **Ans.**

Example 36. *Find the Laplace transform of* $e^{-2t}u_\pi(t)$ *where*

$$u_\pi(t) = \begin{cases} 0; & t<\pi \\ 1; & t>\pi \end{cases}$$

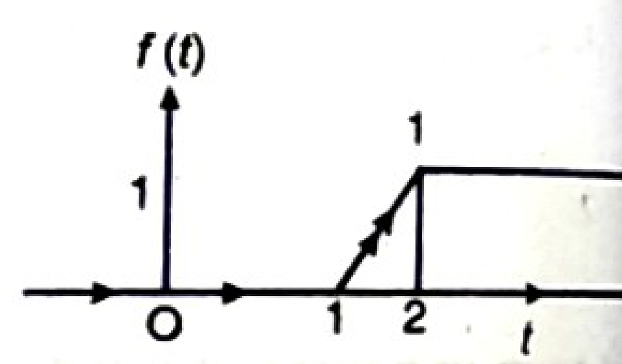
Solution. $u_\pi(t) = \begin{cases} 0; & t<\pi \\ 1; & t>\pi \end{cases}$

$$u_\pi(t) = u(t-\pi)$$
$$L[u_\pi(t)] = L[u(t-\pi)] = \frac{e^{-\pi s}}{s}$$
$$L\left[e^{-2t}u_\pi(t)\right] = \frac{e^{-\pi(s+2)}}{s+2}$$ **Ans.**

Example 37. *Express the following function in terms of unit step function and find its Laplace transform* $f(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & 2<t \end{cases}$ (U.P. II Semester, Summer 2002)

Solution. The above function shown in the figure is expressed in algebraic form

$$f(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & 2<t \end{cases} \quad \ldots(1)$$



$$f(t) = (t-1)\left[u(t-1)-u(t-2)\right]+u(t-2)$$
$$= (t-1)u(t-1)-u(t-2)\{t-1-1\} = (t-1)u(t-1)-(t-2)u(t-2)$$
$$Lf(t) = L(t-1)u(t-1) - L(t-2)u(t-2)$$
$$= \frac{e^{-s}}{s^2}-\frac{e^{-2s}}{s^2}$$ **Ans.**

Example 38. *Represent f(t) = sin 2t, 2π < t < 4π and f(t) = 0 otherwise, in terms of unit step function and then find its Laplace transform.*

Solution. $f(t) = \begin{cases} \sin 2t, & 2\pi<t<4\pi \\ 0, & \text{otherwise} \end{cases}$

$$f(t) = \sin 2t\left[u(t-2\pi)-u(t-4\pi)\right]$$
$$L[f(t)] = L[\sin 2t.u(t-2\pi)] - L[\sin 2t.u(t-4\pi)]$$
$$= e^{-2\pi s}L\sin 2(t+2\pi) - e^{-4\pi s}L\sin 2(t+4\pi)$$
$$= e^{-2\pi s}L\sin 2t - e^{-4\pi s}L\sin(2t)$$

$$= e^{-2\pi s}\frac{2}{s^2+4} - e^{-4\pi s}\frac{2}{s^2+4}$$
$$= \left(e^{-2\pi s}-e^{-4\pi s}\right)\frac{2}{s^2+4}$$ **Ans.**

Example 39. *A function f (t) obeys the equation* $f(t) + 2\int_0^t f(t)\,dt = \cosh 2t$ *Find the Laplace transform of f (t).* (U.P. II Semester Summer 2006)

Solution. We have, $f(t) + 2\int_0^t f(t)\,dt = \cosh 2t$

Taking Laplace transformation of both the sides, we get

$$L\{f(t)\} + 2L\int_0^t f(t)\,dt = L(\cosh 2t) \Rightarrow F(s)+2.\frac{1}{s}F(s) = \frac{s}{s^2-4}$$
$$\Rightarrow F(s)\left\{1+\frac{2}{s}\right\} = \frac{s}{s^2-4} \Rightarrow F(s)\left\{\frac{s+2}{s}\right\} = \frac{s}{s^2-4}$$
$$\Rightarrow F(s) = \left(\frac{s}{s^2-4}\right)\left(\frac{s}{s+2}\right) \Rightarrow F(s) = \frac{s^2}{\left(s^2-4\right)(s+2)}$$ **Ans.**

EXERCISE 10.4

Find the Laplace transform of the following:

1. $f(t) = \begin{cases} t-1, & 1<t<2 \\ 0, & \text{otherwise} \end{cases}$ **Ans.** $\frac{e^{-s}-e^{-2s}}{s^2}-\frac{e^{-2s}}{s}$

2. $e^t u(t-1)$ **Ans.** $\frac{e^{-(s-1)}}{s-1}$

3. $\frac{1-e^{2t}}{s}+tu(t)+\cosh t.\cos t$ **Ans.** $\log\frac{s-2}{s}+\frac{1}{s^2}+\frac{s^3}{s^4+4}$

4. $t^2u(t-2)$ **Ans.** $\frac{e^{-2s}}{s^3}\left(4s^2+4s+2\right)$

5. $\sin t\, u(t-4)$ **Ans.** $\frac{e^{-4s}}{s^2+1}[\cos 4 + s\sin 4]$

6. $f(t) = K(t-2)\left[u(t-2)-u(t-3)\right]$ **Ans.** $\frac{K}{s^2}\left[e^{-2s}-(s+1)e^{-3s}\right]$

7. $f(t) = K\frac{\sin \pi t}{T}\left[u(t-2T)-u(t-3T)\right]$ **Ans.** $\frac{K\pi T}{s^2T^2+\pi^2}\left(e^{-2sT}-e^{-3sT}\right)$

Express the following in terms of unit step functions and obtain Laplace transforms.

8. $f(t) = \begin{cases} t, & 0<t<2 \\ 0, & 2<t \end{cases}$ **Ans.** $u(t)-u(t-2), \frac{1-(2s+1)e^{-2s}}{s^2}$

9. $f(t) = \begin{cases} \sin t, & 0<t<\pi \\ t, & t>\pi \end{cases}$ **Ans.** $\frac{1+e^{-\pi s}}{s^2+1}+\frac{e^{-\pi s}(\pi s+1)}{s^2}$

10. $f(t) = \begin{cases} 4, & 0<t<1 \\ -2, & 0<t<3 \\ 5, & t>3 \end{cases}$ **Ans.** $\frac{4-6e^{-s}+7e^{-3s}}{s}$

10.20 PERIODIC FUNCTIONS

Let $f(t)$ be a periodic function with period T, then

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Proof. $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \ldots$

Substituting $t = u + T$ in second integral and $t = u + 2T$ in third integral, and so on.

$$L[f(t)] = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \ldots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du + \ldots$$

$$[f(u) = f(u+T) = f(u+2T) = f(u+3T) = \ldots]$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \ldots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \ldots\right]\int_0^T e^{-st} f(t)\,dt \qquad \left[1 + a + a^2 + a^3 + \ldots = \frac{1}{1-a}\right]$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt. \qquad \textbf{Proved.}$$

Example 40. *Find the Laplace transform of the waveform*

$$f(t) = \left(\frac{2t}{3}\right), 0 \le t \le 3.$$

Solution. $$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$L\left[\frac{2t}{3}\right] = \frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\left(\frac{2}{3}t\right)dt = \frac{1}{1-e^{-3s}}\,\frac{2}{3}\left[\frac{te^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{2}{3}\,\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right] = \frac{2}{3}\cdot\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s} + \frac{1-e^{-3s}}{s^2}\right]$$

$$= \frac{2e^{-3s}}{-s\left(1-e^{-3s}\right)} + \frac{2}{3s^2} \qquad \textbf{Ans.}$$

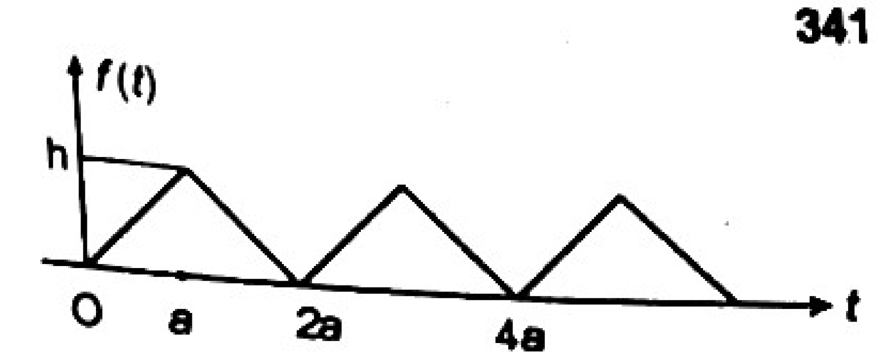
Example 41. *Draw the graph and find the Laplace transform of the following function of period 2a:*

$$f(t) = \begin{cases} \dfrac{h}{a}t, & 0 < t < a \\ \dfrac{h}{a}(2a-t), & a < t < 2a \end{cases}$$

(Uttarakhand, II Semester, GBTU, 2010)

Solution. Period $= 2a = T$

Laplace transform of periodic function $f(t)$



$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

$$\Rightarrow \quad L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

On putting the values of $f(t)$, we get

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\cdot\frac{h}{a}t\,dt + \int_a^{2a} e^{-st}\frac{h}{a}(2a-t)\,dt\right].$$

$$= \frac{h}{a}\left(\frac{1}{1-e^{-2as}}\right)\left[\left\{\frac{te^{-st}}{-s} - 1\cdot\frac{e^{-st}}{(-s)^2}\right\}_0^a + \left\{(2a-t)\frac{e^{-st}}{(-s)} - (-1)\frac{e^{-st}}{(-s)^2}\right\}_a^{2a}\right]$$

$$= \frac{h}{a}\left(\frac{1}{1-e^{-2as}}\right)\left[\left\{\frac{ae^{-as}}{-s} - \frac{e^{-as}}{(-s)^2} - 0 + \frac{1}{s^2}\right\} + \left\{(2a-2a)\frac{e^{-2as}}{(-s)} + \frac{e^{-2as}}{s^2} - \left((2a-a)\frac{e^{-as}}{-s} + \frac{e^{-as}}{s^2}\right)\right\}\right]$$

$$= \frac{h}{a(1-e^{-2as})}\left\{-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right\}$$

$$= \frac{h}{a(1-e^{-2as})}\left\{\frac{1}{s^2}(1 - 2e^{-as} + e^{-2as})\right\}$$

$$= \frac{h(1-e^{-as})^2}{a\,s^2(1+e^{-as})(1-e^{-as})} = \frac{h}{a}\left[\frac{1-e^{-as}}{s^2(1+e^{-as})}\right] \qquad \textbf{Ans.}$$

Example 42. *Draw the graph of the periodic function*

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$$

and find its Laplace transform. *(U.P. Second Semester, 2003)*

Solution. Period $= 2\pi = T$

Laplace transform of Periodic functions

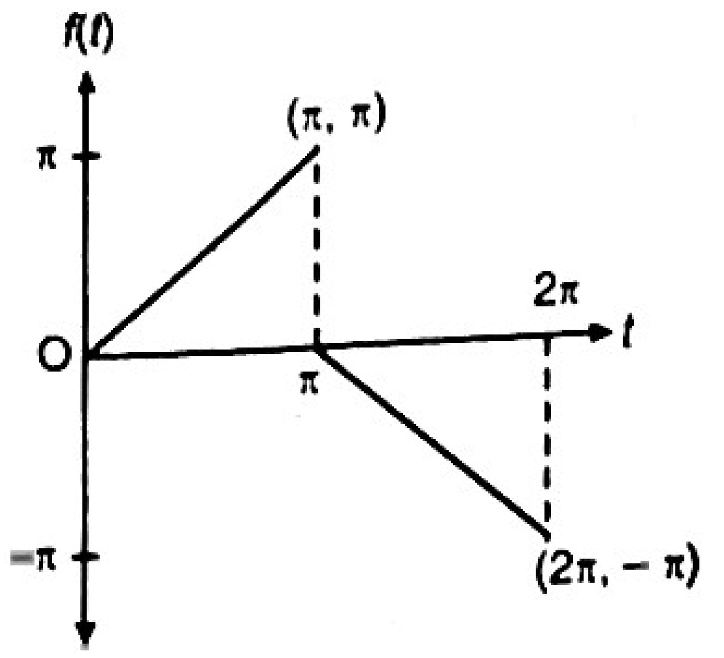
$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2\pi s}}\left[\int_0^\pi e^{-st}t\,dt + \int_\pi^{2\pi} e^{-st}(\pi - t)\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left\{\frac{t.e^{-st}}{-s} - 1.\frac{e^{-st}}{(-s)^2}\right\}_0^\pi + \left\{(\pi - t)\frac{e^{-st}}{(-s)} - (-1)\frac{e^{-st}}{(-s)^2}\right\}_\pi^{2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\left\{\frac{\pi.e^{-\pi s}}{-s} - \frac{e^{-\pi s}}{(-s)^2} - 0 + \frac{1}{s^2}\right\} + \left\{(\pi - 2\pi)\frac{e^{-2\pi s}}{-s} + \frac{e^{-2\pi s}}{s^2} - \left((\pi - \pi)\frac{e^{-\pi s}}{-s} + \frac{e^{-\pi s}}{s^2}\right)\right\}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left\{-\frac{\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \pi\frac{e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - 0 - \frac{e^{-\pi s}}{s^2}\right\}$$



$$=\frac{1}{1-e^{-2\pi s}}\left\{-\frac{\pi}{s}e^{-\pi s}+\frac{\pi}{s}e^{-2\pi s}+\frac{1}{s^2}-\frac{1}{s^2}.e^{-\pi s}+\frac{1}{s^2}e^{-2\pi s}-\frac{e^{-\pi s}}{s^2}\right\}$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\frac{\pi}{s}\left(e^{-2\pi s}-e^{-\pi s}\right)+\frac{1}{s^2}\left(1+e^{-2\pi s}-2e^{-\pi s}\right)\right]=\frac{-\pi s e^{-\pi s}\left(1-e^{-\pi s}\right)+\left(1-e^{-\pi s}\right)^2}{s^2\left(1+e^{-\pi s}\right)\left(1-e^{-\pi s}\right)}$$

$$=\frac{-\pi s e^{-\pi s}+1-e^{-\pi s}}{s^2\left(1+e^{-\pi s}\right)}$$

Ans.

Example 43. *Find the Laplace transform of the function (Half wave rectifier)*

$$f(t)=\begin{cases}\sin\omega t & \text{for } 0<t<\dfrac{\pi}{\omega}\\ 0 & \text{for } \dfrac{\pi}{\omega}<t<\dfrac{2\pi}{\omega}.\end{cases}$$

(U.P. II Semester, 2010, Summer 2002)

Solution. $L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$

$$=\frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{2\pi/\omega}e^{-st}f(t)\,dt \qquad \left[\begin{array}{l}f(t) \text{ is a periodic function}\\ T=\dfrac{2\pi}{\omega}\end{array}\right.$$

$$=\frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega}e^{-st}\sin\omega t\,dt+\int_{\pi/\omega}^{2\pi/\omega}e^{-st}\times 0\times dt\right]$$

$$=\frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\pi/\omega}e^{-st}\sin\,\omega t\;dt$$

$$\left[\int e^{ax}\sin bx\,dx=e^{ax}\frac{(a\sin bx-b\cos bx)}{a^2+b^2}\right]$$

$$L[f(t)]=\frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-st}(-s\sin\omega t-\omega\cos\omega t)}{s^2+\omega^2}\right]_0^{\pi/\omega}$$

$$=\frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{\omega e^{-\frac{\pi}{\omega}s}+\omega}{s^2+\omega^2}\right]=\frac{\omega\left[1+e^{-\frac{\pi s}{\omega}}\right]}{(s^2+\omega^2)\left[1-e^{-\frac{2\pi s}{\omega}}\right]}=\frac{\omega\left[1+e^{-\frac{\pi s}{\omega}}\right]}{(s^2+\omega^2)\left(1-e^{-\frac{\pi s}{\omega}}\right)\left(1+e^{-\frac{\pi s}{\omega}}\right)}$$

$$=\frac{\omega}{\left(s^2+\omega^2\right)\left[1-e^{-\frac{\pi s}{\omega}}\right]}$$

Ans.

Example 44. *Find the Laplace Transform of the Periodic function (saw tooth wave)*

$$f(t)=\frac{kt}{T}\text{ for } 0<t<T,\quad f(t+T)=f(t)$$

Solution. $L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}\frac{kt}{T}dt$

$$=\frac{1}{1-e^{-sT}}\frac{k}{T}\int_0^T e^{-st}.t\,dt=\frac{k}{T\left(1-e^{-sT}\right)}\left[t\frac{e^{-st}}{-s}-\int 1.\frac{e^{-st}}{-s}dt\right]_0^T \quad \text{Integrating by parts}$$

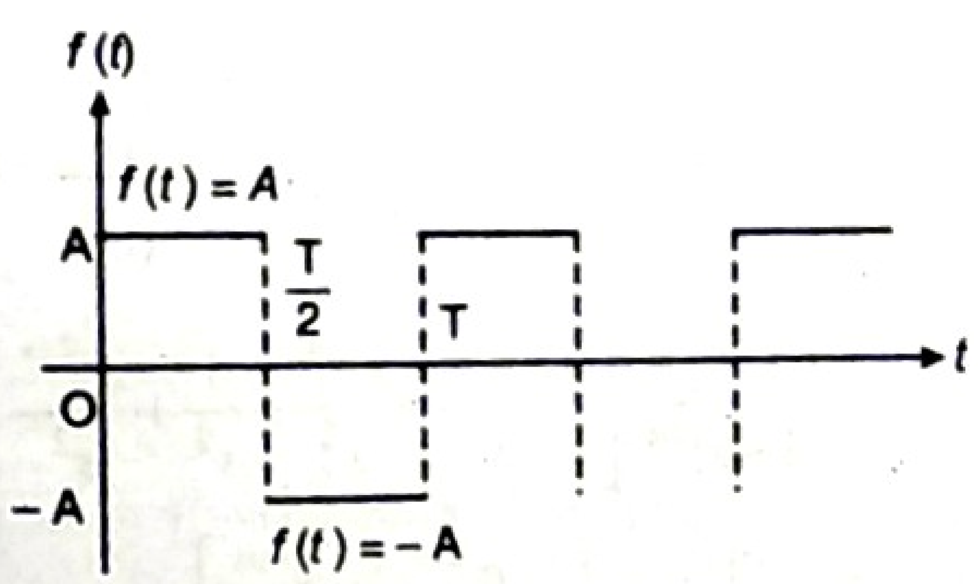
$$=\frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{te^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_0^T=\frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{Te^{-sT}}{-s}-\frac{e^{-sT}}{s^2}+\frac{1}{s^2}\right]$$

$$=\frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{Te^{-sT}}{-s}+\frac{1}{s^2}\left(1-e^{-sT}\right)\right]$$

$$=-\frac{ke^{-sT}}{s\left(1-e^{-sT}\right)}+\frac{k}{Ts^2}$$

Ans.

Example 45. *Obtain Laplace transform of rectangular wave given by*



Solution. We know that Laplace transform of a periodic function *i.e.*,

$$L\,f(t)=\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}=\frac{\int_0^{\frac{T}{2}}e^{-st}A\,dt+\int_{\frac{T}{2}}^{T}e^{-st}(-A)\,dt}{1-e^{-sT}}$$

$$=A\frac{\left[\frac{e^{-st}}{-s}\right]_0^{\frac{T}{2}}-\left[\frac{e^{-st}}{-s}\right]_{\frac{T}{2}}^{T}}{1-e^{-sT}}=\frac{A}{1-e^{-sT}}\left[-\frac{e^{-\frac{sT}{2}}}{s}+\frac{1}{s}+\frac{e^{-sT}}{s}-\frac{e^{-\frac{sT}{2}}}{s}\right]$$

$$=\frac{A}{s\left(1-e^{-sT}\right)}\left[1-2e^{-\frac{sT}{2}}+e^{-sT}\right]=\frac{A}{s\left(1-e^{-sT}\right)}\left[1-e^{-\frac{sT}{2}}\right]^2$$

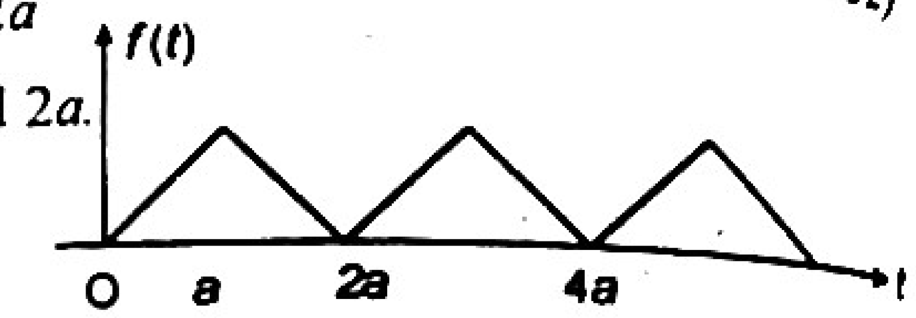
$$= \frac{A\left[1-e^{-\frac{sT}{2}}\right]^2}{s\left(1+e^{-\frac{sT}{2}}\right)\left(1-e^{-\frac{sT}{2}}\right)} = \frac{A}{s}\frac{\left(1-e^{-\frac{sT}{2}}\right)}{\left(1+e^{-\frac{sT}{2}}\right)}$$

$$= \frac{A\left(e^{\frac{sT}{4}}-e^{-\frac{sT}{4}}\right)}{s\left(e^{\frac{sT}{4}}+e^{-\frac{sT}{4}}\right)} = \frac{A}{s}\tanh\frac{sT}{4}$$ **Ans.**

Example 46. *Draw the graph of the following periodic function and find its Laplace transform:*

$$f(t)=\begin{cases} t & \text{for } 0<t\le a \\ 2a-t & \text{for } a<t<2a \end{cases}$$ *(U.P. II Semester, Summer 2002)*

Solution. The given function is periodic with period $2a$.



$$\therefore \quad L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a f(t)e^{-st}dt + \int_a^{2a} f(t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a t\, e^{-st}dt + \int_a^{2a}(2a-t)\, e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left\{t\frac{e^{-st}}{-s}-\frac{e^{-st}}{(-s)^2}\right\}_0^a + \left\{(2a-t)\frac{e^{-st}}{-s}+\frac{e^{-st}}{(-s)^2}\right\}_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[-\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2}+\frac{e^{-2as}}{s^2}-2\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{1}{\left(1-e^{-2as}\right)}\left(1+e^{-2as}-2e^{-as}\right)$$

$$= \frac{1}{s^2}\frac{\left(1-e^{-as}\right)^2}{\left(1+e^{-as}\right)\left(1-e^{-as}\right)} = \frac{1}{s^2}\left[\frac{1-e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{1}{s^2}\left[\frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{e^{\frac{as}{2}}+e^{-\frac{as}{2}}}\right] = \frac{1}{s^2}\tanh\frac{as}{2}$$ **Ans.**

Example 47. *A periodic square wave function f(t), in terms of unit step functions, is written as*

$$f(t)=k\left[u_0(t)-2u_a(t)+2u_{2a}(t)-2u_{3a}(t)+\ldots\right]$$

Show that the Laplace transform of f (t) is given by

$$L\left[f(t)\right]=\frac{k}{s}\tanh\left(\frac{as}{2}\right)$$

Solution.

$$f(t)=k\left[u_0(t)-2u_a(t)+2u_{2a}(t)-2u_{3a}(t)+\ldots\right]$$

$$f(t) = k\,[u\,(t-0) - 2u\,(t-a) + 2u\,(t-2a) - 2u\,(t-3a) + \ldots\ldots]$$

$$L\,[f(t)] = k\,[L\,u\,(t-0) - 2\,L\,u\,(t-a) + 2\,L\,u\,(t-2a) - 2\,L\,u\,(t-3a) + \ldots..]$$

$$= k\left[\frac{1}{s}-2\frac{e^{-as}}{s}+2\frac{e^{-2as}}{s}-2\frac{e^{-3as}}{s}+\ldots\right]$$

$$= \frac{k}{s}\left[1-2e^{-as}+2e^{-2as}-2e^{-3as}+\ldots\right]$$

$$= \frac{k}{s}\left[1-2(e^{-as}-e^{-2as}+e^{-3as}-\ldots)\right]$$ $\left[\begin{array}{l}\text{This is G.P.}\\ \text{Sum}=\frac{a}{1-r}\end{array}\right]$

$$= \frac{k}{s}\left[1-2\frac{e^{-as}}{1+e^{-as}}\right] = \frac{k}{s}\left[\frac{1+e^{-as}-2e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{k}{s}\left[\frac{1-e^{-as}}{1+e^{-as}}\right] = \frac{k}{s}\left[\frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{e^{\frac{as}{2}}+e^{-\frac{as}{2}}}\right] = \frac{k}{s}\tanh\frac{as}{2}$$ **Ans.**

EXERCISE 10.5

1. Find the Laplace transform of the periodic function

   $f(t)=e^t$ for $0<t<2\pi$ **Ans.** $\dfrac{e^{2(1-s)\pi}-1}{(1-s)\left(1-e^{-2\pi s}\right)}$

2. Obtain Laplace transform of full wave rectified sine wave given by

   $f(t)=\sin\omega t,\quad 0<t<\dfrac{\pi}{w}$ **Ans.** $\dfrac{\omega}{\left(s^2+\omega^2\right)}\coth\dfrac{\pi s}{2\omega}$

3. Find the Laplace transform of the staircase function

   $f(t)=kn,\ np<t<(n+1)p,\ n=0,1,2,3$ **Ans.** $\dfrac{ke^{ps}}{s\left(1-e^{-ps}\right)}$

Find Laplace transform of the following:

4. $f(t)=t^2,\ 0<t<2,\ f(t+2)=f(t)$ **Ans.** $\dfrac{2-e^{-2s}-4se^{-2s}-4s^2e^{-2s}}{s^3\left(1-e^{-2s}\right)}$

5. $f(t)=\begin{cases}1, & 0\le t\le\frac{a}{2}\\ -1, & \frac{a}{2}\le t<a\end{cases}$ *(U.P. II Semester, 2004)* **Ans.** $\dfrac{1}{s}\tanh\dfrac{as}{4}$

6. $f(t)=\begin{cases}\cos\omega t, & 0<t<\frac{\pi}{\omega}\\ 0, & \frac{\pi}{\omega}<t<\frac{2\pi}{\omega}\end{cases}$ **Ans.** $\dfrac{s}{\left(s^2+\omega^2\right)\left(1-e^{-\frac{\pi s}{\omega}}\right)}$

7. $f(t)=\begin{cases} t, & 0<t<1 \\ 0, & 1<t<2 \end{cases}$ $f(t+2)=f(t)$ **Ans.** $\dfrac{1-e^{-s}(s+1)}{s^2\left(1-e^{-2s}\right)}$

8. $f(t)=\begin{cases} \dfrac{2t}{T}, & 0\le t\le \dfrac{T}{2} \\ \dfrac{2}{T}(T-t), & \dfrac{T}{2}\le t\le T \end{cases}$ $f(t+T)=f(t)$ **Ans.** $\dfrac{2}{Ts^2}\tanh\dfrac{sT}{4}-\dfrac{1}{s\left(e^{\frac{sT}{2}}+1\right)}$

EXERCISE 10.7

Evaluate the following by using Laplace Transform:

1. $\int_0^\infty t e^{-4t} \sin t \, dt$ **Ans.** $\frac{8}{289}$

2. $\int_0^\infty \frac{e^{-2t} \sinh t \sin t}{t} dt$ **Ans.** $\frac{1}{2}\tan^{-1}\frac{1}{2}$

3. $\int_0^\infty \frac{\sin^2 t}{t^2} dt$ **Ans.** $i\frac{5}{2}$

4. $\int_0^\infty \frac{e^{-t} - e^{-4t}}{t} dt$ **Ans.** log 4

10.24 FORMULATION OF LAPLACE TRANSFORM

S.No.	$f(t)$	$F(s)$
1.	e^{at}	$\frac{1}{s-a}$
2.	t^n	$\frac{\lceil n+1}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$
3.	sin at	$\frac{a}{s^2+a^2}$
4.	cos at	$\frac{s}{s^2+a^2}$
5.	sinh at	$\frac{a}{s^2-a^2}$
6.	cosh at	$\frac{s}{s^2-a^2}$
7.	$u(t-a)$	$\frac{e^{-as}}{s}$
8.	$\delta(t-a)$	e^{-as}
9.	e^{bt} sin at	$\frac{a}{(s-b)^2+a^2}$
10.	e^{bt} cos at	$\frac{s-b}{(s-b)^2+a^2}$
11.	$\frac{t}{2a}\sin at$	$\frac{s}{(s^2+a^2)^2}$
12.	t cos at	$\frac{s^2-a^2}{(s^2+a^2)^2}$
13.	$\frac{1}{2a^3}(\sin at - at\cos at)$	$\frac{1}{(s^2+a^2)^2}$
14.	$\frac{1}{2a}(\sin at + at\cos at)$	$\frac{s^2}{(s^2+a^2)^2}$

10.25 PROPERTIES OF LAPLACE TRANSFORM

S.No.	Property	f(t)	F (s)
1.	Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$, $a > 0$
2.	Derivative	$\frac{df(t)}{dt}$ $\frac{d^2f(t)}{dt^2}$ $\frac{d^3f(t)}{dt^3}$	$sF(s) - f(0)$, $s > 0$ $s^2F(s) - sf(0) - f'(0)$, $s > 0$ $s^3F(s) - s^2f(0) - sf'(0) - f''(0)$, $s > 0$
3.	Integral	$\int_0^t f(t)\,dt$	$\frac{1}{s}F(s)$, $s > 0$
4.	Initial Value	$\lim_{t\to 0} f(t)$	$\lim_{s\to\infty} sF(s)$
5.	Final Value	$\lim_{t\to\infty} f(t)$	$\lim_{s\to\infty} sF(s)$
6.	First shifting	$e^{-at}f(t)$	$F(s+a)$
7.	Second shifting	$f(t)\,u(t-a)$	$e^{-a}\,L\,f(t+a)$
8.	Multiplication by t	$t\,f(t)$	$-\frac{d}{ds}F(s)$
9.	Multiplication by t^n	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}F(s)$
10.	Division by t	$\frac{1}{t}f(t)$	$\int_s^\infty F(s)\,ds$
11.	Periodic function	$f(t)$	$\frac{\int_0^T e^{-st}f(t)}{1-e^{-st}}$ $f(t+T) = f(t)$
12.	Convolution	$f(t) * g(t)$	$F(s)\,G(s)$